l. The following values are critical:

$$\sin^2\lambda = l \qquad \text{when } \beta = 0,\pi$$

$$\sin^2\beta = (-l/3h) \qquad \text{when } \lambda = 0,\pi$$

$$\sin^2\lambda = [(3h + l)/(3h + 1)] \qquad \text{when } \beta = \pm\pi/2$$

Thus for $-\frac{1}{3} < h < 0$ the contours appear as in Fig. 2 with the regions of libration interior to the $l = -3h$ contours. For $-1 < h < -\frac{1}{3}$ the contours appear as in Fig. 3 with the regions of libration interior to the $l = 1$ contours.

References

[1] Wintner, A., *The Analytical Foundations of Celestial Mechanics* (Princeton University Press, Princeton, N. J., 1947), Secs. 442, 443, 471, 472.

[2] La Salle, J. and Lefschetz, S., *Stability by Liapunov's Direct Method* (Academic Press Inc., New York, 1961), pp. 28–38.

Oscillations of a Fluid in a Rectilinear Conical Container

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In a recent publication, Troesch[1] notes that very few exact solutions are known for the free oscillations of an ideal (incompressible, nonviscous, irrotational) fluid in axially symmetric containers. He refers to the cylinder and paraboloid treated in Lamb;[2] however, the special case of a rectilinear conical container is attributed to Levin as an unpublished memorandum. Although the closed-form solution (for the first eigenvalue) now can be obtained quite easily from the comprehensive material presented by Troesch, it is of interest to record the initial special case that gave rise to the subsequent generalizations. Furthermore, Troesch's approach proceeds from a general series expansion followed by an examination of the results to determine what class of problems has been solved. The present method is direct in that the geometry of the given problem dictates the choice of a suitable coordinate system.

Consider a conical tank at rest with semivertex angle $\alpha = \pi/4$ as shown in Fig. 1. The linearized hydrodynamic equations are

$$\nabla^2\phi = 0 \text{ throughout the fluid} \quad (1)$$

$$\partial\phi/\partial n = 0 \text{ on the tank walls} \quad (2)$$

$$(\partial^2\phi/\partial t^2) + g(\partial\phi/\partial y) = 0 \text{ on the free surface } y = h \quad (3)$$

where ϕ is the scalar velocity potential. Using cylindrical coordinates [with $r = (x^2 + z^2)^{1/2}$] and a trial form for ϕ given by

$$\phi = F(t)G(r,y)\cos\theta \quad (4)$$

leads to

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right) + \frac{\partial^2 G}{\partial y^2} - \frac{G}{r^2} = 0 \quad (5)$$

The change of variables $W = r + y$ and $V = r - y$ results in coordinate surfaces for the tank walls. Equation (5) becomes

$$(W^2 + 2WV + V^2)[G_{WW} + G_{VV}] + (W + V)[G_W + G_V] - 2G = 0 \quad (6)$$

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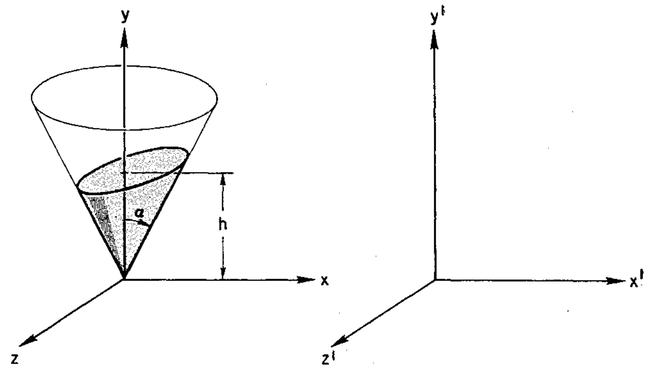

Fig. 1 Tank geometry

and condition (2) becomes

$$\partial G/\partial V = 0 \text{ on } V = 0 \quad (7)$$

A trial expansion for G in the form

$$G = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_{mn} W^m V^n$$

when substituted into Eq. (6) leads to double recursion relations for the A_{ij}. In order to satisfy these relations and Eq. (7), it is necessary that all coefficients A_{ij} vanish except $A_{20} = -A_{02}$; hence

$$G = A_{20}(W^2 - V^2) \qquad G = Kry \quad (8)$$

Thus,

$$\phi = F(t)ry\cos\theta \quad (9)$$

This solution fulfills conditions (1) and (2). Condition (3) leads to

$$hr\ddot{F}(t) + rgF(t) = 0$$
$$F(t) = A\cos(\omega t + \lambda) \quad (10)$$

where $\omega = (g/h)^{1/2}$ represents the (first) natural frequency of the system.

If the tank is not at rest but subjected to rectilinear accelerations, condition (3) is replaced by

$$(\partial^2\phi/\partial t^2) - A_y(\partial\phi/\partial y) + \dot{A}_x x = 0 \text{ on the free surface } y = h \quad (3')$$

where A_x and A_y are the accelerations of the tank with respect to an inertial coordinate system (x',y',z'). The time dependence of the velocity potential now is given by

$$h\ddot{F}(t) - A_y(t)F(t) + \dot{A}_x(t) = 0$$

For example, a periodic forced excitation in the x direction leads to

$$h\ddot{F}(t) + gF(t) = B\cos\alpha t \qquad \alpha \neq \omega$$

$$F = A\cos(\omega t + \lambda) + [B\cos\alpha t/(g - h\alpha^2)]$$

The first term actually is damped for a real fluid, and consequently the motion is represented by the second term. The resonance as $\alpha \to \omega = (g/h)^{1/2}$ has been verified experimentally.

References

[1] Troesch, B. A., "Free oscillations of a fluid in a container," *Boundary Problems in Differential Equations*, edited by R. E. Langer (University of Wisconsin Press, Madison, Wis., 1960), pp. 279–299.

[2] Lamb, H., *Hydrodynamics* (Dover Publications Inc., New York, 1945), 6th ed., Chap. VIII.